FILTRATION LAW WITH QUADRATIC NONLINEARITY

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A quadratic filtration law which generalizes the Darcy and Brinkman laws is considered.

The Darcy law

$$p_{,i} = -\frac{\mu}{\varkappa} u_i \tag{1}$$

is usually used for macroscopic description of slow flows of viscous liquids and gases through porous media.

Equation (1) agrees well with experiment for flows which change slowly over space, at moderate pressure heads and filtration velocities. But when these conditions are not satisfied deviations from the Darcy law are observed. This can occur, for example, in face regions of wells with anomalously high stratal pressure or in filtration of a liquid (gas) through thin membranes (at significant pressure heads). Another question regarding the applicability of Eq. (1) arises in describing the contact between a liquid-filled porous medium and a pure liquid. Since the Darcy equation does not contain derivatives of the velocity with respect to the coordinates (i.e., is a zeroth order equation), and the Navier–Stokes equations (second-order equations) are usually used to describe microflows of liquid within the pore space and flows of the pure liquid outside the porous medium, complications can occur in specifying the boundary conditions on the interphase boundary. Similar problems also occur in consideration of filtration of multiphase or multicomponent liquids.

To eliminate these shortcomings and achieve agreement with experiment various modifications of the Darcy law have been proposed. Thus, to describe filtration at significant velocities and pressure heads nonlinear filtration laws of the form

$$p_{,i} = -\frac{\mu}{\varkappa} u_i g\left(\frac{|\mathbf{u}|\rho d}{\mu}, m\right),$$

have been used (see, for example, [1]), where d is the characteristic linear dimension of the micromotion: $|\mathbf{u}|\rho d/\mu = \mathrm{Re}_f$ is the Reynolds number of the filtration micromotion; g is a dimensionless "influence" function. Expanding g in a series in $\mathbf{u}$ up to the second power, we obtain the often used quadratic

$$p_{,i} = -\frac{\mu}{\varkappa} u_i - \frac{\beta}{\sqrt{\varkappa}} \rho |\mathbf{u}| u_i, \tag{2}$$

where it is assumed that $d \sim \sqrt{\kappa}$ [1].

For the case where shear stresses in the filtering liquid cannot be neglected, Brinkman proposed a filtration law (for an incompressible liquid) in the form [2]

$$p_{,i} = -\frac{\mu'}{\varkappa} u_i + \mu' u_{i,jj}, \tag{3}$$

where the parameter μ' has the sense of an effective liquid viscosity in the porous medium, and generally does not coincide with μ. Equation (3) is similar to the equation of motion (in the Stokes approximation with renormalized viscosity μ') of a conventional linear-viscous liquid, on which additional mass forces $f_i = -\alpha' u_i$, $\alpha' = \mu'/\kappa$ act. Such a liquid is sometimes called a "Brinkman liquid" [3].

Equation (3) contains second derivatives of the velocity with respect to coordinates, and therefore eliminates the complications involving boundary conditions referred to above.

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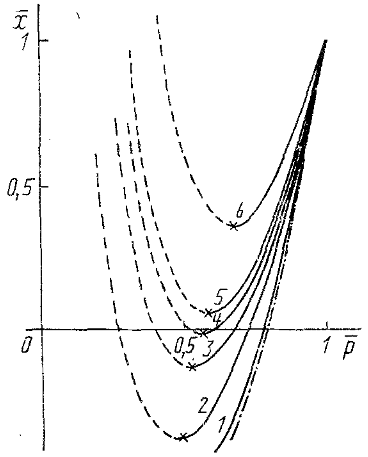

Fig. 1. Dimensionless pressure $\bar{p}$ vs dimensionless distance $\bar{x}$ (Q = 0.01; a = 20; dash–dot line, Darcy law): 1) b = 10; 2) 1000; 3) 1500; 4) 1800; 5) 2000; 6) 3000.

If S is the surface separating the porous medium and the pure liquid, then the boundary conditions on S have the usual form [3]:

$$[n_i u_i] = 0, \quad [t_i u_i] = 0,$$

where n_i and t_i are unit vectors normal and tangent to S, and the square brackets indicate a step in the quantity contained therein upon transition through the surface S. In addition the conditions

$$[\sigma_{ij} t_i n_j] = 0, \quad [\sigma_{ij} n_i n_j] = 0,$$

must be satisfied on S, where $\sigma_{ij} = -p\delta_{ij} + \mu'(u_{i,j} + u_{j,i})$ is the "viscous stress" tensor for a Brinkman liquid.

We obtain a filtration law having the properties of Eqs. (2) and (3).

We will limit our examination to isotropic, homogeneous, and nondeforming porous media, and for simplicity we will consider steady state flows. We will use the approach proposed in [3], but will consider the liquid compressible.

We will consider the steady-state Navier–Stokes equation (which we assume describes microflows of the liquid within the pore space)

$$\rho u_j u_{i,j} + p_{,i} = \left(\eta + \frac{1}{3}\mu\right) u_{j,ji} + \mu u_{i,jj}. \tag{4}$$

We take the Fourier transform of Eq (4) for the pore space only, obtaining

$$R_i(\mathbf{k}) = -\left(\eta + \frac{1}{3}\mu\right) k_i k_j u_j(\mathbf{k}) - \mu k_j k_j u_i(\mathbf{k}) - G_i[\mathbf{k};\ \mathbf{u}], \tag{5}$$

where $R_i(\mathbf{k})$ is the Fourier transform of the left side of Eq. (4) and $G_i[\mathbf{k}, \mathbf{u}]$ is an operator which appears as a result of considering conditions at infinity and the boundary conditions on the surface of the pores. We will limit ourselves to the case where the operator G_i is quadratic in $\mathbf{u}$, i.e.,

$$G_i[\mathbf{k};\ \mathbf{u}] = A_i[\mathbf{k};\ \mathbf{u}] + B_i[\mathbf{k};\ \mathbf{u},\ \mathbf{u}] = \int A_{ij}(\mathbf{k},\ \mathbf{k}')\, u_j(\mathbf{k}')\, d\mathbf{k}' + \iint B_{ijl}(\mathbf{k},\ \mathbf{k}',\ \mathbf{k}'')\, u_j(\mathbf{k}')\, u_l(\mathbf{k}'')\, d\mathbf{k}' d\mathbf{k}''. \tag{6}$$

Since the medium is assumed homogeneous, then the operators A_i and B_i must be invariant relative to shear, consequently [4]:

$$A_{ij}(\mathbf{k},\ \mathbf{k}') = \tilde{A}_{ij}(\mathbf{k})\, \delta(\mathbf{k} - \mathbf{k}'), \quad B_{ijl}(\mathbf{k},\ \mathbf{k}',\ \mathbf{k}'') = \tilde{B}_{ijl}(\mathbf{k}',\ \mathbf{k}'')\, \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}''), \tag{7}$$

while $\tilde{B}_{ij\ell}$ must satisfy symmetry conditions for the simultaneous replacement $j \leftrightarrow \ell$, $k' \leftrightarrow k''$.

We will assume that $\tilde{A}_{ij}$ and $\tilde{B}_{ij\ell}$ are polynomials in $\mathbf{k}$ of no higher than second order (this corresponds to the situation where in the defining filtration equations in x-space there are no derivatives of order higher than second). Considering the isotropic nature of the porous medium, we may write $\tilde{A}_{ij}$, $\tilde{B}_{ij\ell}$ in the form

$$\tilde{A}_{ij}(\mathbf{k}) = \alpha_0 \delta_{ij} - \alpha_1 \mathbf{k}^2 \delta_{ij} + \alpha_2 k_i k_j, \quad \tilde{B}_{ijl}(\mathbf{k}',\ \mathbf{k}'') = i\beta_1 (k_i' + k_i'')\, \delta_{jl} + \frac{i\beta_2}{2} (k_j' \delta_{il} + k_l'' \delta_{ij}) + \frac{i\beta_3}{2} (k_j'' \delta_{il} + k_l' \delta_{ij}). \tag{8}$$

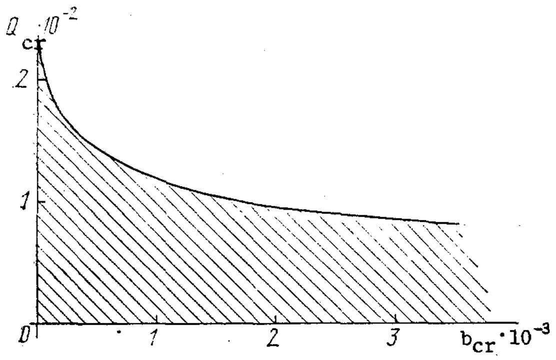

Fig. 2

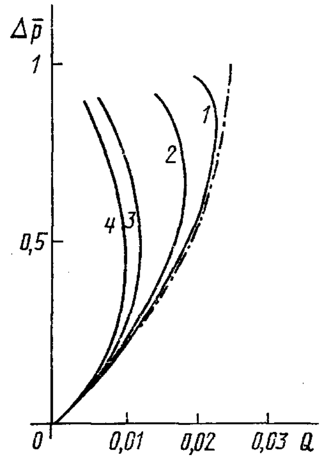

Fig. 3

Fig. 2. Quantity Q_{cr} vs b_{cr} (a = 20).

Fig. 3. Indicator curves $\overline{\Delta p} \sim Q$ for various values of parameter b ($\overline{\Delta p} = \bar{p}(\bar{x} = 1) - \bar{p}(\bar{x} = 0)$, a = 20): 1) b = 10; 2) 100; 3) 1000; 4) 2000; dash-dot line, Darcy law.

Using Eqs. (5)-(8) and an inverse Fourier transform, from Eq. (5) we obtain

$$\rho' u_j u_{i,j} + p_{,i} = \left(\eta' + \frac{1}{3}\mu'\right) u_{j,ji} + \mu' u_{i,jj} - \alpha_0 u_i - \beta_1 (u_j u_j)_{,i} - \beta_2 (u_{j,j}) u_i, \tag{9}$$

where $\rho' = \rho + \beta_3$, $\mu' = \mu - \alpha_1$, $\eta' = \eta + 1/3\ \alpha_1 + \alpha_2$.

If in Eq. (9) we neglect the second derivatives with respect to coordinates and the squares of the velocities, we obtain the conventional Darcy law. Considering the second derivatives, but as before neglecting the squares of the velocities, we arrive at a Brinkman type filtration equation with renormalized viscosity coefficients. Finally, omitting from Eq. (9) terms containing second derivatives, we obtain filtration equations with quadratic corrections to the Darcy law:

$$\rho' u_j u_{i,j} + p_{,i} = -(\alpha_0 + \beta_2 u_{j,j}) u_i - \beta_1 (u_j u_j)_{,i}. \tag{10}$$

In contrast to Eq. (2) the dependence on u in the right side of Eq. (10) is analytic.

Unfortunately the extreme complexity of the pore space geometry produces almost insurmountable difficulties for an exact analytical calculation of the coefficients α_i and β_i, and therefore they must be determined from experiment. It is possible to obtain analytic results only for rarefied periodic structures or "weak solutions." In [3] the question of calculation of the renormalized viscosity μ' for Eq. (3) was considered. It developed that for suspensions $\mu' > \mu$, while for media of high porosity ($m \sim 1$) $\mu' < \mu$, while the parameter η' as a rule, is greater than η [5].

As an example of application of the modified Darcy law, we will consider steady state one-dimensional filtration of an ideal gas. We take the filtration law in the form of Eq. (10) and supplement it with equations of continuity and state:

$$p_{,x} = -\alpha_0 u - \beta u_{,x} u - \rho u u_{,x}, \quad (\rho u)_{,x} = 0, \quad p = c^2 \rho,$$

where $\beta = 2\beta_1 + \beta_2 + \beta_3$; c is the speed of sound in the gas. Denoting $\rho u = q$ (where q is the flow rate), we find that p satisfies the equation

$$\left(\frac{p^2}{2}\right)_{,x} - \beta q^2 c^4 \frac{p_{,x}}{p^2} - q^2 c^2 \frac{p_{,x}}{p} = \alpha_0 c^2 q. \tag{11}$$

Equation (11) with the boundary condition $p(x_0) = p_0$ is easily integrated. In dimensionless notation its solution has the form

$$\frac{1}{2}(\bar{p}^2 - 1) + Q^2 b \left(\frac{1}{\bar{p}} - 1\right) - Q^2 \ln \bar{p} = aQ(\bar{x} - 1), \tag{12}$$

where $Q = qc/p_0$, $\bar{p} = p/p_0$, $\bar{x} = x/x_0$, $b = \beta c^2/p_0$, $a = \alpha_0 c x_0/p_0$. In this case the classical Darcy law, Eq. (1), leads to a solution

$$\frac{1}{2}(\bar{p}^2-1)=aQ(\bar{x}-1). \quad (13)$$

In Fig. 1 the dash–dot line shows the solution of Eq. (13), while the solid lines correspond to Eq. (12) for fixed a, Q and various (increasing from bottom to top) values of the parameter b, the dashed segments indicating "nonphysical" branches of the $\bar{p}(\bar{x})$ curves.

Let us assume that the problem under consideration models influx of gas into a "well" located at the point $\bar{x} = 0$. It follows from Fig. 1 that operation of such a well is not possible at all b values. For a given Q there exists a b_{cr}, such that at $b > b_{cr}$ filtration ceases. Since we are considering steady state problems, this simply means that for a given Q a steady state filtration process is impossible if $b > b_{cr}(Q)$. Figure 2 shows the dependence of b_{cr} on Q. The range of parameters in which steady state filtration is possible is shaded.

Figure 3 shows indicator curves of the "well" considered, giving the dependence of flow rate Q on pressure head $\Delta\bar{p} = \bar{p}(\bar{x} = 1) - \bar{p}(\bar{x} = 0)$ for fixed a and various b. The dash–dot line shows the corresponding dependence for the Darcy law. The characteristic bend of the indicator curves in the figure has been observed in practice and the experiments of [6], and has usually been related to deformability of the porous media (petroleum and gas collectors). The example presented shows that such behavior of the dependence of $\Delta\bar{p}$ on Q is also possible in nondeforming media, if the filtration law deviates from the Darcy law.

NOTATION

p, pressure; μ, shear viscosity of liquid (gas); η, bulk viscosity; κ, permeability; u, flow velocity; m, porosity; ρ, liquid (gas) density; f_i, mass forces; S, surface separating liquid and porous medium; c, speed of sound; σ_{ij}, viscous stress tensor; G_i, A_i, B_i, operators with respect to u; A_{ij}, $B_{ij\ell}$, $\tilde{A}_{ij}$, $\tilde{B}_{ij\ell}$, kernels and regularized kernels of operators A_i, B_i; α'_i, β_i, β, parameters; k_i, k'_i, k''_i, Fourier transform parameters; ρ', μ', η', renormalized density, shear, and bulk viscosity; q, flow rate; x, distance; p_0, initial pressure at point x_0; a, b, positive parameters; $\bar{p}$, $\bar{x}$, Q, dimensionless p, x, and q; b_{cr}, critical value of parameter b; $\Delta\bar{p}$, dimensionless pressure head.

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